Optimal Land Use Switching Policy

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Abstract

In this paper, we examine the real option to switching problems. We evaluate a situation where farmers can switch between different crops in response to crop price changes. We contribute to the literature in a variety of respects. First, in conventional models farmers can always choose to convert between different land uses at a cost. We extend the existing literature by adding a time cost as well as a cash cost. While pure cost models may be appropriate when farmers switch from farming sheep to cows and vice versa, in some activities, such as planting, this is not necessarily true, as there is a waiting time period as there is a delay in yields. Farmers in our example do not only face a cash cost but also face a prolonged period of no production until new crops reach maturity. Second, given the long conversion periods, our model allows farmers to reverse their switches. Given the complexity of the decision and thus the model, we employ computational methods to investigate how various factors enter into the decision making process.

Key Words: Real Options, Optimal Switch, ADI Method, Investment, Stochastic Prices, Dynamic Modelling.

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1 Introduction

Real options valuation is an important area of research that is used in evaluating projects in the real financial world and investment decisions under uncertainty. There is a broad range of literature that use this technique to solve managerial choices problem in different industries (for example, mining, electricity, metallurgical, and steel). We use this technique to evaluate a land allocation problem. The literature on land use falls into two main categories: real options pricing framework to real estate and agriculture. We examine the option to switch between different crops in response to crop price changes, which falls in the second category. Land allocation problems often incorporate a cash cost but lack the time dimension. Therefore, including such a cost would resemble a real options problem so we will adopt this approach.

Financial options valuation looks into an option that gives a payoff based on a stock price and calculate how much it is worth in any period. In contrast, the real options problem implies that there is not only a situation we run into optionality but also quite often in the real financial world we might find ourselves in a situation where we can choose to exercise options. We evaluate one of these situations in this paper, which is the option to reverse land use at any time. However, there is not any specific market to trade real options. Therefore, they have a relative subjective value instead of a market value. The term *real options* was coined by Myers (1977) and it refers to available options in an investment. There is a wide range of possible options in a project namely the option to defer, abandon, expand, contract, and switch. Real options analysis is widely used by scholars to evaluate investment decisions (see, for example, McDonald and Siegel (1985a,b), Brennan and Schwartz (1985), Brekke and Oksendal (1994), Slade (2001), and Samis and Davis (2014)). The incentive of having options is to have a limited loss but unlimited profit and the more volatile options are the more exciting they are for investors.

We contribute to the literature in a variety of aspects. One of the common assumptions of real options is irreversibility. Dixit (1989, 1992) and Dixit and Pindyck (1994) examine the irreversible investment under uncertainty to evaluate industry entry and exit. In our model, farmers have this option to switch from planting one crop to another crop. Farmers can always choose to convert between the different land uses, but they have to pay the cost of switching so they get the chance to reverse their choice. We extend the existing literature by adding a time cost as well as a cash cost. Pure cost models may be appropriate when farmers switch from farming sheep to cows and vice versa. However, in some activities, such as planting, it is not necessarily true as there is a waiting time period. Farmers in our example not only face a cash cost but also face a prolonged period of no production while new crop (e.g. saplings) reach maturity. This is one of the contributions of our model that takes into consideration time cost and the opportunity cost along with price cost. In other words, we evaluate the interaction between uncertainty, timing, and switching cost at the same time in order to find the optimal policy for farmers.

There is fairly scarce studies about real option to switch in agriculture. Song, Zhao, and Swinton (2011) use a real option framework to evaluate farmers decision to switch between traditional annual crops and perennial energy crops. Their results show that the stochastic returns and government polices affect on the optimal switching policy. Our model differs from their model in several aspects. First, the structure of our model is fundamentally different and more advanced by exploiting the ADI technique (see Section 2.2). Second, our model takes into consideration the time cost while gives farmers the possibility to reverse their decision. Third, we incorporate the correlation between two crops in the model to capture its impact throughout the decision making process instead of evaluating this factor in a sensitivity analysis process. In this regard, we apply Clewlow and Strickland (1998) approach to account the correlation between two crops. We explain the method and calculation in Section 2.2 in more details.

Hendricks, Smith, and Sumner (2014), evaluate the switching between corn and soybean in a conceptual framework when the corresponding price changes. Results show that the responds to price shocks in the short run is greater than in the long run. However, their study does not fall in the real options category and follows an econometric analysis of field data. Their findings help to have a better understanding of farmers' heterogeneous incentives and responses to price shocks as well as crops rotations. Some studies evaluate the land conversion problem by setting up an experiment to capture the impact of underlying factors that drive farmers' decisions. For example, Ihli, Maart-Noelck, and Musshoff (2013) evaluate the farmers' behaviour in a within-subject designed experiment. Their findings show that the classical investment theory and real options approach may not explain the exact behaviour of farmers. However, their study is designed in an abstract and laboratory environment, which is different from real world experiment and actual business setting. Their study provides some possible explanations for farmers' behaviour not only with economic factors but also by taking into account the socio-demographic and farm-specific variables.

As we mentioned earlier, a part of real options literature in agricultural land use are focused on irreversible investments and entry/exit strategies. For instance, Shah and Ando (2016) examine the choice between conserving or converting land in a case study in Indonesia. Farmers in this study have the option to conserve the land and earn carbon credit or convert the land to palm oil and have the return of it. They analyse the financial and ecological cost of temporary and permanent land conservation policy incentives, which leads to policy advices. Also, Cunha-e-sa and Franco (2016) evaluate an irreversible real option problem in land redevelopment. They evaluate the residential rent changes and its impact on preserved open areas nearby. This study advices that boundaries such as development halt period and minimum-lot-size can postpone land conversion and helping to protect open spaces. Similarly, there is a wide range of earlier studies in this context, which use real options for agricultural land use optimization (see Seyoum-Tegegn and Chan (2013), Cyr, Hanagriffb, and Kwongc (2010), Tauer (2006), Khanna, Isik, and WinterNelson (2000), Tegene, Wiebe, and Kuhn (1999), Price and Wetzstein (1999), Winter-Nelson and Amegbeto (1998), Purvis, Boggess, Moss, and Holt (1995)). According to underlying assumptions and set up, this strand of theoretical and empirical studies are different from our study.

The remainder of this paper proceeds as follows: In Section 2 we describe our methodology of solving Black-Scholes partial differential equation in order to find the optimal land value with different crops. In this regard, we introduce the pricing method, ADI Crank-Nicolson application, boundary conditions, and finally cost functions. Section 3 presents the baseline parameter values and a range of scenarios to analyse. As this study is theoretical by nature that is motivated by farming, we generate the underlying crop prices and parameters using simulation and bootstrap techniques. Section 4 evaluates our numerical results by analysing the optimal exercise frontiers and then sensitivity analysis according to underlying variables: cost, time, volatility, drift, discount rate, and correlation. Finally, in Section 5, we summarize our results and express our advices for policy makers, practitioners, and feature research.

2 Methodology

We assume a scenario that farmers have the option to plant two type of crops: crop 1 and crop 2. We use the term state 1 for being active with crop 1 and state 2 for being active with crop 2. When farmers are active with planting crop 1 and the price of crop 2 increases, farmers get this incentive to switch from crop 1 to crop 2 and vice versa. In other words, farmers have the option to switch from one state to another state as the price of alternative crop changes. Also, we assume a transitional state in between that farmers wait there until they become an active player in the alternative state that we call it *Fallow* state. However, this waiting period can be very fast and in a snap of fingers or it might be long enough that cause farmers to lose the cash flow that they could have been received with being active in one of the activities. The longer the waiting period the more the time cost. This scenario creates a problem of real option to switch. In other words, in fallow state, the land is vacant and ready for any type of switching which makes this state to embraces option to pause, continue, revert, extend, or contract.

We evaluate this real option problem in three evolutionary models. The reason is to start from a core model, which is the simplest form, and then incorporating more factors into the model. In this way, we can compare and contrast all three models to see how proposed solutions and policies vary according to the corresponding assumptions underlying each model.

Model 1 evaluates a situation where farmers have the choice of being in two different states. Farmers in each state run a different activity and they switch according to how profitable an activity is. However, farmers are supposed to pay a fixed cost to switch. In this model, switching is a snap decision and farmers need to do the cost-benefit analysis to check if it is worth to pay the cost and convert the current activity. Model 2 solves the same problem but if switching is not a snap decision, then there is potential profits to be lost during this period. We calculate these profits and deduct as economic costs to be added to the physical cost of switching. In doing so, we do not keep track of the lost time. In particular, the alternative option or activity should be so profitable to cover the fixed cost and opportunity cost at the same time. Finally, Model 3, similar to Model 2, accounts for the time cost, which implies lost potential profits during switching. Different from Model 2, Model 3 allows farmers to be in a third state referred to as a fallow state.

In this fallow state, farmers have the option to revise their decision and switch back to their original use. Model 3, evaluates the problem of switching in a dynamic form and farmers can switch at any time, given that any cost is computed. In this model there are two activities (Crop 1 and Crop 2) but there are three states (state 1, Fallow, State 2). Farmers can always change their mind anywhere in between and exceed out of the process and go back to the previous state if they pay the time cost and price cost.

2.1 Pricing Method

In the following we illustrate the derivation of pricing method. The underlying assumption about asset prices is that it follows a geometric Brownian motion (GBM). In the context of land use, returns evolve over time due to technological advances and improvements from farming process to storing and transporting to countries worldwide, which in turn enhance the productivity. In apple fruit industry, for example, achieve the benefit of greater productivity farmers by choosing the efficient rootstocks for their orchard. Accordingly, our prices of two competing crops, S_1 and S_2, follow the GBM, which can be defined by the following stochastic differential equation (SDE):

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dz_1 \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dz_2 \end{aligned} \tag{1}$$

where dS_i represents the changes in price level, μ_i is the crop drift, S_i is the crop price, dt is the small interval of time, σ_i is the crop volatility, and finally dz_i is the change in random variable z that is called a Wiener or Brownian motion process.

We implement the two dimensional Black-Scholes equation for the corresponding option pricing of our crops. We apply Ito's lemma on Black-Scholes Partial Differential

Equation (PDE), which gives:

$$\frac{\partial C_i}{\partial t} = \mu_1 S_1 \frac{\partial C_i}{\partial S_1} + \mu_2 S_2 \frac{\partial C_i}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 C_i}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 C_i}{\partial S_2^2} + \rho\sigma_1 S_1 \sigma_2 S_2 \frac{\partial^2 C_i}{\partial S_1 \partial S_2} - rC_i + S_i \quad (2)$$

where $\frac{\partial C_i}{\partial t}$ is the rate of change of land value with respect to time, μ_i and S_i are the expected growth rate and prices of each crop and $\frac{\partial C_i}{\partial S_i}$ is the rate of change of land value with respect to crop prices, σ_i is the volatility of crop prices and $\frac{1}{2}\sigma_i^2 S_i^2 \frac{\partial^2 C_i}{\partial S_i^2}$ is the diffusive part for each crop, ρ is the correlation between two crops and the term $\rho\sigma_1 S_1 \sigma_2 S_2 \frac{\partial^2 C_i}{\partial S_1 \partial S_2}$ captures the effect of two crops with together, r is the discount rate and rC_i captures the convective effect, finally Si is an extra term compared with BS model that we receive as a cash flow from each crop.

In our problem we have an extra term S_i that is the cash flow we receive in each activity. We use Alternating Direction Implicit (ADI) method to solve numerically Black-Scholes PDE model. This method let us to have N lots of $N \times N$ equations instead of $N^k \times N^k$, where k is the number of states or dimensions.

2.2 ADI Crank-Nicolson Application

The numerical solution for a one-dimensional time-depended PDE includes a large linear system of equations. Therefore, the numerical approximation of multidimensional problem will be more complicated than one-dimensional problem as it is dependent on the history of solutions to find the current value. The ADI method, which is a fractional method, use time discretization to solve a PDE of d-dimensional spacial problem. This idea was proposed and extended for the first time by Peaceman and Rachford (1955) and Douglas and Rachford (1955). They used this method to solve the heat equation, which is a parabolic PDE problem. One of the advantages of using ADI method is that it takes advantage of implicit spatial discretization (Crank-Nicolson). In other words, it provides a fast solving tridiagonal matrices, which is a representative of a set of equations.

Before we can apply the ADI method to equation (2) we must transform it into the form of a standard diffusion equation. In this respect, we transform it into the natural logarithms of the asset prices $x_1 = ln(S_1), x_2 = ln(S_2)$, which gives the following PDE.

$$\frac{\partial C_i}{\partial t} = \nu_1 \frac{\partial C_i}{\partial x_1} + \nu_2 \frac{\partial C_i}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 C_i}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 C_i}{\partial x_2^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 C_i}{\partial x_1 \partial x_2} - rC_i + e^{x_i} \tag{3}$$

where ν_1 is $(\mu_1 - \frac{1}{2}\sigma_1^2)$ and ν_2 is $(\mu_2 - \frac{1}{2}\sigma_2^2)$.

The ADI method is a powerful way of solving multidimensional PDEs specifically in option valuation. However, in option pricing, we might have correlation between the underlying assets that ADI initially is not developed to handle this situation. Therefore, in order to solve equation (3), we need to use a transformation to remove the correlation part. Clewlow and Strickland (1998) propose to compute the eigenvectors and eigenvalues of the covariance matrix of x_1 and x_2.

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \tag{4}$$

where on the left hand, we have covariance matrix and on the right hand: eigenvectors, eigenvalues, and the transpose of eigenvectors. This method provides an uncorrelated linear combinations of x_1 and x_2. We use the eigenvectors to transform the PDE (3) to equation (5).

$$\frac{\partial C_i}{\partial t} = \alpha_1 \frac{\partial C_i}{\partial y_1} + \alpha_2 \frac{\partial C_i}{\partial y_2} + \frac{1}{2}\lambda_1 \frac{\partial^2 C_i}{\partial y_1^2} + \frac{1}{2}\lambda_2 \frac{\partial^2 C_i}{\partial y_2^2} - rC_i + e^{(e_{i1}y_1 + e_{i2}y_2)} \qquad i = 1, 2, ..., n \tag{5}$$

The ADI method also has this drawback that is not able to handle the zeroth

(rC_i) and first order terms $\left(\alpha_1 \frac{\partial C_i}{\partial y_1} + \alpha_2 \frac{\partial C_i}{\partial y_2}\right)$ very well. However, we can use the Finite Difference Method (FDM) discretization of a first and second order scheme with second order accuracy (Tavella and Randall, 2000).

$$\begin{aligned} \frac{\partial u}{\partial x} &\simeq \frac{u(x+\Delta x, y) - u(x-\Delta x, y)}{2\Delta x} \\ \frac{\partial^2 u}{\partial x^2} &\simeq \frac{u(x+\Delta x, y) - 2u(x,y) + u(x-\Delta x, y)}{\Delta x^2} \end{aligned} \tag{6}$$

We exploit this discretization to find a system of equations for each direction in ADI method. In equation (5), we can infer that the cash flow of each crop can be computed as follows:

$$CF_i = e^{(e_{i1}y_1 + e_{i2}y_2)} \tag{7}$$

where CF_i is the cash flow of crop i, e_{ij} are the eigenvectors, and y_i is the transformed log price of crop i. Consequently, we can construct the constraint of the option pricing model as follows:

$$C_i \geq C_j - K_{ji} \tag{8}$$

where C_i is the land value from crop i, C_j is the land value from crop j, and K_{ji} is the cash cost to pay and switch from planting crop j to crop i.

2.3 Boundary Conditions

Boundary conditions are necessary to solve PDE problems. The choice of imposing wrong boundary conditions might lead either to an incorrect solution or diverging and model explosion. While there is the possibility that boundary conditions are not in line with the system of equations, we set our lower and upper boundaries conditions based on the finite difference discretization of $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$, which are made of second order schemes. The set up is as follows:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &\approx \frac{-u(x+2\Delta x, y) + 4u(x+\Delta x, y) - 3u(x,y)}{2\Delta x} \\
\frac{\partial u}{\partial x} &\approx \frac{3u(x,y) - 4u(x-\Delta x, y) + u(x-2\Delta x, y)}{2\Delta x} \\
\frac{\partial^2 u}{\partial x^2} &\approx \frac{-u(x+3\Delta x, y) + 4u(x+2\Delta x, y) - 5u(x+\Delta x, y) + 2u(x,y)}{\Delta x^2} \\
\frac{\partial^2 u}{\partial x^2} &\approx \frac{2u(x,y) - 5u(x-\Delta x, y) + 4u(x-2\Delta x, y) - u(x-3\Delta x, y)}{\Delta x^2}
\end{aligned} \tag{9}
$$

Equation (9) includes the corresponding lower and upper boundaries to use for ADI calculation. We use these boundaries along with our main system of equations (6) to find the optimal solution. These boundaries can be derived by expanding the function evaluation in Taylor series, which provide the derivative approximation (see Tavella and Randall (2000)).

2.4 Cost Functions

As we mentioned earlier, our analysis breaks down into three evolutionary models. In this part, we explain how to calculate the cost for each model. Model 1 is the simplest model that farmers pay a fixed cost and switch to the alternative crop without waiting. This model is the common model in the literature, which does not take into consideration the time cost and opportunity cost. We set up the cost function for this model in two different ways. The simple cost function is to consider a specific cash cost K amount. The complicated cost function is to calculate the exact amount of cost accordingly. We calculate the complicate cost by solving the following equation:

$$K_{ji} = \int_{0}^{T_{i,1}} K_{i,1}e^{-rt}dt + \int_{T_{i,1}}^{T_{i,1}+T_{j,2}} K_{j,2}e^{-rt}dt \tag{10}$$

where, K_{ji} is the amount of corresponding cash cost to switch from state i to state j through the available switching option or vice versa. $T_{i,1}$ is the time to uproot a crop in state i, and $T_{j,2}$ is the time to plant a crop in state j. Discount rate r is the primary factor we use to measure the time value of money in any small incremental change of time dt.

The cost of switching is not only the amount of cash cost but also missing the current cash flow, which comes from sticking to the current activity. Therefore, the opportunity cost is an extra cost that farmers should bare along with cash cost. We set up Model 2 to measure this extra cost. However, Model 2 does not take into consideration the time cost as there is no waiting. We set up the expected opportunity cost function as follows:

$$K_{ji} = \frac{S_i}{\mu_i - r}\{e^{(\mu_i - r)T} - 1\} \tag{11}$$

where, K is the opportunity cost, S_i is the amount of missing cash flow, μ_i is the drift, r is the discount rate, and finally T is the time of switching. This calculation gives a constant cost to pay for the opportunity cost. Therefore, in Model 2 we add the fixed cost from equation (10) with opportunity cost from equation (11) to find the total cash cost of switching.

$$K_{ji} = \int_{0}^{T_{i,1}} K_{i,1}e^{-rt}dt + \int_{T_{i,1}}^{T_{i,1}+T_{j,2}} K_{j,2}e^{-rt}dt + \frac{S_i}{\mu_i - r}\{e^{(\mu_i - r)T} - 1\} \tag{12}$$

Finally, Model 3 is the comprehensive model that embraces Model 1 and Model 2 as well as taking into consideration the time cost. This model optimizes the land allocation in a dynamic process through time. In Model 3, farmers uproot the current crop and switch to a fallow state, which is a vacant land without any crop and wander as long as they see any benefit to switch to one of the crops. As a follow up, they have this option to go back to the previous state if they change their mind or move forward and plant a new crop. We calculate the corresponding cost inside the model through the ADI process. In other words, we solve equation (5), but instead of having a cash flow we should pay the corresponding cost of uprooting and planting.

$$\frac{\partial C_i}{\partial t} = \alpha_1 \frac{\partial C_i}{\partial y_1} + \alpha_2 \frac{\partial C_i}{\partial y_2} + \frac{1}{2}\lambda_1 \frac{\partial^2 C_i}{\partial y_1^2} + \frac{1}{2}\lambda_2 \frac{\partial^2 C_i}{\partial y_2^2} - rC_i - K_{ij} \qquad i = 1, 2, ..., n \tag{13}$$

3 Data and parameter values

3.1 Baseline Parameters and Scenarios

The primary focus of this paper is on building a theoretical model and we test our theory by proposing some baseline parameters. In this regard, Table 1 presents a set of sensible numbers to start with our optimization of land allocation. In Section 4, we examine the sensitivity analysis and the effect of change in each variable on switching. In other words, we evaluate how changes in input parameters might make one way of switching system more interesting or uninteresting for farmers.

We connect models with different types of real options according to their corresponding parameters. In turn, the difference between models, changes when the time cost varies from one type of farming to another. For example, in animal farming (like sheep and beef) there is almost no time cost and we expect all three models to be the same while in seasonal crops they have a time cost close to one year and in orchard (like

TABLE 1: *Benchmark and Sensitivity Analysis Scenarios*

Note 1*: Variable are scaled based on thousands (1000) per hectare and time variables are annually.*
Note 2*: The sign + indicates that an increase in the variable causes the switching to increase; - indicates that an increase in the variable causes the switching to decrease, and ? indicates that the effect is not clear.*

Parameters	**Symbol**	**Crop 1**	**Crop 2**	**Effect**
Panel A: Baseline Variables				
Cost (uproot,plant)	K_{ij}	(1,10)	(1,10)	
Time (uproot,plant)	T_{ij}	(1,3)	(1,3)	
Volatility	σ_i	0.10	0.12	
Drift	μ_i	0.01	0.01	
Discount rate	r	0.05		
Correlation	ρ	0.3		
Panel B: Sensitivity Analysis				
Scenario 1: Change in Cost and Time				
Cost (uproot,plant)	K_{ij}	(10,100)	(10,100)	-
Time (uproot,plant)	T_{ij}	(3,8)	(3,8)	-
Scenario 2: Change in Volatility and Drift				
Volatility	σ_i	0.16	0.18	?
Drift	μ_i	0.04	0.04	-
Scenario 3: Change in Discount Rate				
Discount rate	r	0.12		-
Scenario 4: Change in Correlation				
Correlation	ρ	+ 0.8 - 0.8		?

apple and grape) the time cost increases to 5 or 6 years, which is a crucial time cost and we expect that all three models to deviate from each other. Our modelling approach allows us to investigate these different cases under one general framework and to carry out counterfactual analysis so with regression models and very unlikely to evaluate different scenarios.

According to our baseline parameters in Table 1, it costs \$1000 and takes 1 year to uproot each crop whereas it costs \$10000 and takes 3 years to plant. Crop 1 has a

10% volatility with 1% drift while crop 2 has 12% volatility with 1% drift, which means farmers pay the cost out of their pocket and there is 1% return from the crop. These two crops have a correlation of 0.3 in a market with 5% discount rate. In Section 4, we use these parameters to solve our model for the value of land, extract the optimal exercise frontier, and finally examine the frequency of switch in three evolutionary models. Panel B in Table 1 shows four different scenarios that we will examine in Section 4.2 in details. In each scenario we will change one parameter at a time and evaluate the impact on the farmers switching behaviour.

3.2 Simulation and Bootstrap

In terms of crop prices we generate a simulation of 65000 paths over 200 years for two competing crops. The choice of 65000 paths and 200 years is ad hoc and can be any number but large enough to utilize the law of large number. As we increase the number of simulations we find a closer proxy for the prices of the industry. This price simulation helps examine farmers' switching behaviour and evaluate when a threshold is crossed to trigger a switch and find when they hit the barriers and how many times they change the state of activity. We use equation (14) to generate the crop prices process.

$$
\begin{aligned}
\Delta \log S_1 &= (\mu_1 - \frac{\sigma_1^2}{2})\Delta t + \sigma_1 \epsilon_1 \\
\Delta \log S_2 &= (\mu_2 - \frac{\sigma_2^2}{2})\Delta t + \sigma_2(\rho_{12}\epsilon_1 + \sqrt{1-\rho_{12}^2}\epsilon_2)
\end{aligned}
\tag{14}
$$

where $\Delta \log S_i$ is the change in log price of each crop, μ_i is the drift or cash flow growth of each crop, σ_i is the volatility of each crop, Δt is the time step, ρ is the correlation between two crops, and ϵ_i is the normal random innovation. The ϵ_1 and ϵ_2 are independent and are normally distributed with a mean and variance of $N(0, \sqrt{\Delta t})$.

Another useful technique that we exploit through our analysis is bootstrapping. We use this technique as robustness test to examine whether our results across models holds when we repeat the same experiment for so many times, which makes it close to real world problem. Therefore, we solve our optimization problem according to baseline parameters and their changes in a sensitivity analysis manner (see Section 4). The corresponding analysis are based on generated data that we get from 65000 simulation paths in 200 years. In this regard, we find the switching pattern of farmers in different scenarios. We generate 10000 random samples with replacement based on our main results in each scenario, which has the same size of 65000 observations. This resampling provides enough simulation to carry out hypothesis testing including the difference between mean of switch, probability of one or more switch, and probability of two or more switch across our models in different scenarios.

4 Numerical Results

4.1 Optimal Exercise Frontiers

In this section, for a given set of parameters, we solve the valuation problem to find how much our land is worth, apply the strategic implication to calculate the optimal strategy, and finally, implement a numerical application to see how often farmers switch between crops in a life time. In this regard, we calibrate our proposed theory with the set of sensible numbers from Table 1 to quantify land redevelopment in three evolutionary models: Model 1, Model 2, and Model 3. The difference between three models arises from the complexity of calculating switching cost as the price of each crop changes over the course of time. According to the set up and constraints, each model will have a different optimal exercise frontier. Optimal frontiers help us to count how often farmers hit the barrier to exercise their option and switch from one crop to another crop. Figure 1 compares the optimal exercise frontiers for three models according to baseline parameters in Table 1.

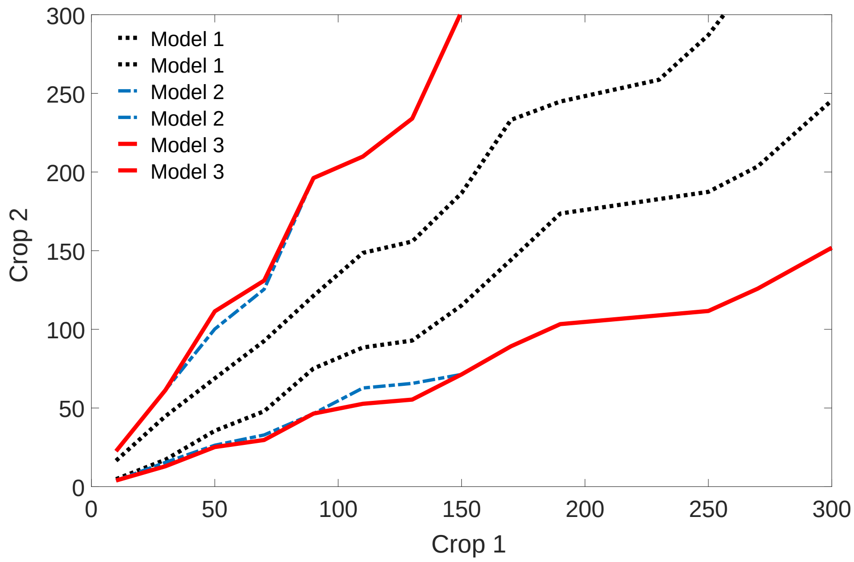

FIGURE 1: Optimal Exercise Frontier for Baseline Parameters

Figure 1 displays Model 1 in dotted lines. This model is oversimplified and farmers would hit the barriers very often because they need to pay only a certain amount of cash cost to switch. In addition, farmers do not change their switching policy with change in crop values because they look at the relative prices. Comparatively, Model 2 with dashed-dotted lines provides less chance for farmers to switch due to the opportunity cost and missing out the cash flow of the existing crop. In other words, farmers should not only pay the cash cost in Model 1 but also pay some of the cost that is partially towards the time cost, which is related to the current crop value. This opportunity cost is so cheap for crops that are not worth much but very expensive for crops that are lucrative. Finally, Model 3 in solid lines fully and formally takes into account time cost and has exercise frontiers with a wider gap. Model 3, compared with the other two models, makes it even harder to switch due to the extra time cost compared with Model 1 and full time cost compared with Model 2 that incorporates partial time cost. Farmers in Model 3 have this chance to abort the process and go back to what they were growing and not to wait for the full year. However, they have to pay a cash cost and wait for a while to resume their initial activity. Comparing all three models show that time cost is proportional to crop values whereas the cash cost does not scale according to crop prices. Having said that, we notice that the optimal exercise frontiers in all three models are close to each other when crop prices are low but further apart when the crop prices are high. It is important to know that any change in underlying parameters will cause the optimal exercise frontiers to get closer or move apart. As the time increases more the gap between Model 2 and Model 3 increases more, which is not significant in our baseline parameters.

The jaggedness of the optimal exercise frontiers is due to the discreteness in our grids. In other words, we discretize the PDE problem and solve it with a specific time step, which is not continuous. Therefore, as we increase the number of steps or make our discrete points closer we will achieve smoother lines. The caveat is that the higher number of time step along with ADI method and dimensionality slow down the speed of solving the valuation problem. Having said that, in some cases, for example, when we have negative correlation we will achieve smooth line rather than a gagged line. The main reason of jaggedness is the use of transformation, which is implementing eigenvalue and eigenvectors on log prices to remove correlation between crops so that we could use ADI method. The negative correlation affects the covariance matrix and results in smooth line due to different transformation process. Figure 2 is an example of this situation and in accordance with negative correlation sensitivity analysis in Section 4.2.4.

Figure 2 shows optimal exercise frontier for three different models: Model 1 with dot line, Model 2 with dash-dot line, and finally Model 3 with solid line.

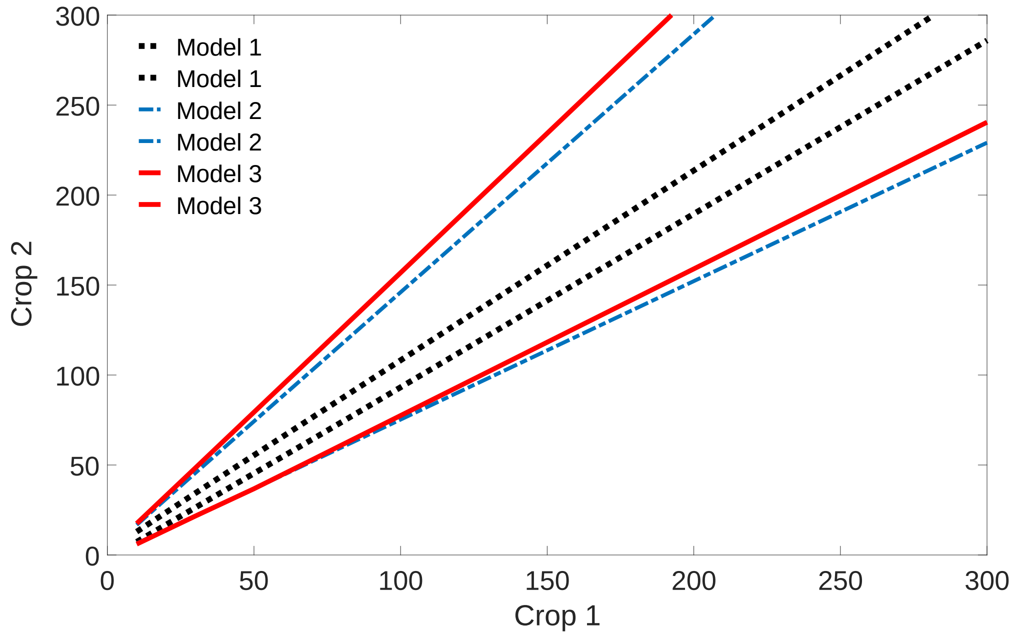

FIGURE 2: Optimal Exercise Frontier for Baseline Parameters (Negative Correlation)

4.2 Sensitivity Analysis

In this section, we analyse the likelihood of switching when the baseline parameters change in Table 1. Therefore, we evaluate different scenarios by having simulated GBM paths of crop prices from equation (14). We begin our analysis by evaluating the behaviour of farmers according to the baseline parameters from Table 1. Farmers in our simulation start planting crop 1 by being in state 1 and by change in crop prices they switch their state of activity. These results provide a benchmark for further analysis when the underlying parameters change.

TABLE 2: *Benchmark Switching Analysis*

Note *: This table presents the mean of switch, the probability of one or more switch, and two or more switch based on 65000 simulation paths over 200 years. The superscripts ***, **, and * indicate significance at the 1%, 5%, and 10% level respectively.*

Variable	Model 1	Model 2	Model 3
Baseline Parameters Analysis			
Mean	1.6049	1.0072***	0.9621***
P $(n \geqslant 1)$	0.8233	0.7551***	0.7425***
P $(n \geqslant 2)$	0.3914	0.2011***	0.1804***

Table 2 shows the mean of switch, probability of one or more switch, and finally two or more switch based on 65000 simulation paths over 200 ears by farmers. We evaluate the switching behaviour between two competing crops in three different models. In this regard, there will be some combinations where there is no switch from crop 1 to crop 2 nor from crop 2 to crop 1. Also, there are some simulations where there is a switch from crop 1 to crop 2 but not a switch from crop 2 to crop 1. Finally, there are some simulations where there are multiple switching back and forth.

Table 2 reports that the mean of switch, probability of one or more switch, and probability of two or more switch between Model 1 and Model 2 as well as between Model 2 and Model 3 is qualitatively different at the 1% level. However, the difference between Model 1 and Model 2 is larger than the difference between Model 2 and Model 3. The difference arises due to the measure of cost effect in each model. Model 1 is a simplified set up that people traditionally use and it takes into account only the cash cost of switching whereas Model 2 considers cash cost along with an approximation of time cost and Model 3 considers the cash cost along with the exact time cost. In other words, Model 1 captures only the cash cost whereas Model 2 and Model 3 not only capture the cash cost but also the time cost, however, with different accuracy. In the following, we evaluate the change in our baseline parameters, the effect on switch frequency, and the underlying reasons for our results.

4.2.1 Cost and Time Sensitivity Analysis

In this section, we evaluate the impact of an increase in cost and time. In this regard, we increase the cost and time of uprooting from \$1000 and 1 year to \$10000 and 3 years respectively. Also, we increase the cost and time of planting from \$10000 and 3 years to \$100000 and 8 years respectively. Table 3 demonstrates the mean number of switches, probability of one or more switch, and probability of two or more switch across our three models due to an increase in cost and time.

TABLE 3: *Cost and Time Switching Analysis*

Note *: This table presents the mean of switch, the probability of one or more switch, and probability of two or more switch based on 65000 simulation paths over 200 years. The choice of 200 is ad hoc and it should be enough to evaluate the behaviour of farmers. The superscripts ***, **, and * indicate significance at the 1%, 5%, and 10% level respectively.*

Variable	**Model 1**	**Model 2**	**Model 3**
Baseline Parameters Analysis			
Mean	1.6049	1.0072***	0.9621***
P ($n \geqslant 1$)	0.8233	0.7551***	0.7425***
P ($n \geqslant 2$)	0.3914	0.2011***	0.1804***
Scenario 1.A: Change in Cost			
Mean	0.6342	0.5674***	0.4765***
P (n$\geqslant$ 1)	0.5484	0.5225***	0.4524***
P ($n \geqslant 2$)	0.0702	0.0412***	0.0227***
Scenario 1.B: Change in Time			
Mean	1.1582	0.6480***	0.5330***
P ($n \geqslant 1$)	0.7383	0.5895***	0.5100***
P ($n \geqslant 2$)	0.2506	0.0551***	0.0225***

Table 3 reports the baseline parameter results as a benchmark, Scenario 1.A as a sensitivity to change in cost, and Scenario 1.B as a sensitivity to change in time. When we increase the cost of switching we expect that all three models to be shocked by this effect, because all three models take into account the cash cost. Therefore, we notice that mean of switch, probability of one or more switch, and probability of two or more switch decrease compared with our benchmark results. In other words, an increase in cost makes the switching expensive and pulls everybody back, which in result provides less switching. The difference between Model 1 and Model 2 as well as the difference between Model 2 and Model 3 is significant at the 1% level but the magnitude of difference is not very large. However, in Scenario 1.B, once we increase time cost (duration of switch) we notice that the difference of pair models starts to pull apart a lot more than what we have in cost change.

Scenario 1.B reports that an increase in time of switching decreases the mean of switch, probability of one or more switch, and probability of two or more switch in all three models compared to our results in benchmark. The difference across our models are qualitatively different at the 1% level. However, we expect that an increase in time effect provides a larger difference between our competing models when time is a much bigger part of the cost. In this case, Model 2 is different from Model 1 due to larger time cost and Model 3 is different from Model 2 due to formal treatment of time, which becomes more important when time is of the essence. Model 2 implies that an increase in time cost leads to a higher approximation of opportunity cost that farmers should pay along with the cash cost, which depends to what state they are active. Whereas Model 3 implicitly captures the opportunity cost by letting farmers to be in "Fallow" state physically. The more the length of time the more the difference arises between Model 2 and Model 3. Overall, an increase in cost and time has a negative impact on the mean and probability of switching by farmers that the level of change and its magnitude varies according to the scenario and type of model.

4.2.2 Volatility and Drift Sensitivity Analysis

In this section, we change volatility and drift values as the drivers of price process. Therefore, we evaluate the impact of each variable separately. In Scenario 2.A we increase the volatility of crop 1 from 10% to 16% and crop 2 from 12% to 18% and examine the results. In Scenario 2.B, we set back everything and increase the drift of crop 1 from 1% to 4% and crop 2 from 1% to 4% and see what is the impact of drift this time. Table 4 presents the results of this change and the corresponding impact on the mean, probability of one or more switch, and probability of two or more switch.

Table 4: *Volatility and Drift Switching Analysis*

Note 1*: This table presents the mean of switch, the probability of one or more switch, and two or more switch based on 65000 simulation paths over 200 years. The superscripts ***, **, and * indicate significance at the 1%, 5%, and 10% level respectively.*

Variable	**Model 1**	**Model 2**	**Model 3**
Baseline Parameters Analysis			
Mean	1.6049	1.0072***	0.9621***
P ($n \geqslant 1$)	0.8233	0.7551***	0.7425***
P ($n \geqslant 2$)	0.3914	0.2011***	0.1804***
Scenario 2.A: Change in Volatility			
Mean	1.5342	0.9465***	0.9416
P ($n \geqslant 1$)	0.8107	0.6938***	0.6938
P ($n \geqslant 2$)	0.3597	0.1920***	0.1941
Scenario 2.B: Change in Drift			
Mean	2.4251	0.9016***	0.7171***
P ($n \geqslant 1$)	0.8606	0.6862***	0.5949***
P ($n \geqslant 2$)	0.5320	0.1729***	0.1067***

Table 4 shows the baseline parameter results as the benchmark, Scenario 2.A as a sensitivity of change in volatility, and Scenario 2.B as a sensitivity of change in drift. Scenario 2.A reports that an increase in volatility has a negative impact on the three measures of our interest. The results across our models show that the difference between Model 1 and Model 2 is large and significant at the 1% level whereas the difference between Model 2 and Model 3 is almost the same and not significant. In this case, our results suggest that Mode 2 is an approximate of Model 3. Model 2 captures the impact of an increase in volatility by measuring the expected change of the crop prices as an extra cost that decrease switching compared with Model 1. Whereas, Model 3 measures the volatility as an uncertainty in crop prices that has a negative impact on switching. Farmers in Model 3 are concerned that crop prices might move against their favour while they are in the process of uprooting and planting or when they abort the process half way through to go back to the existing activity.

An increase in volatility causes a higher level of uncertainty for farmers to switch. However, as uncertainty goes up, the chance of finding an option to switch also goes up. A higher volatility provides more option value and in results farmers are more reluctant to exercise their option. In other words, we are better off in Model 3 because we get more value from uncertainty but the value of option is so high that makes farmers more reluctant to pull the trigger and exercise their option. As Model 3 is the closest model to reality, we focus on its land value for the time being. Therefore, we display the land value when farmers grow crop 1 in figure 3.

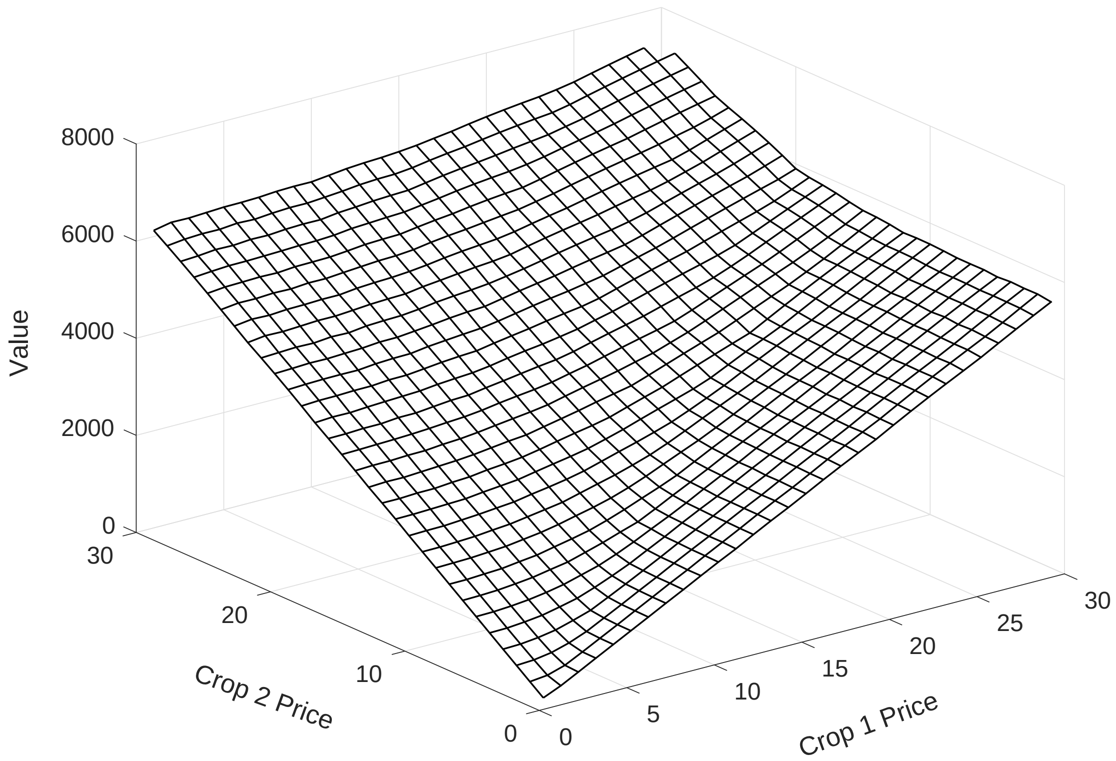

FIGURE 3: Land Value with Two Crops (Model 3)

Figure 3 demonstrates the land value with crop 1 in Model 3. The land value increases in both dimensions because either the crop that farmers are growing is worth more or the option to switch to the competing crop is growing in value. We observe that the land value is curved in the middle due to the time value of the option to switch. Figure 3 shows that the land value is tilted on the side of crop 1 axes that a sign of option to switch from crop 2 to crop 1. If we consider the land value for the other crop we expect that the figure should be tilted the other way around because crop 2 becomes important. In other words, the land value is convex because there is option value. To better illustrate this convexity attribute in figure 3, we plot the diagonal of land value before and after change in volatility for both lands, which is displayed in figure 4.

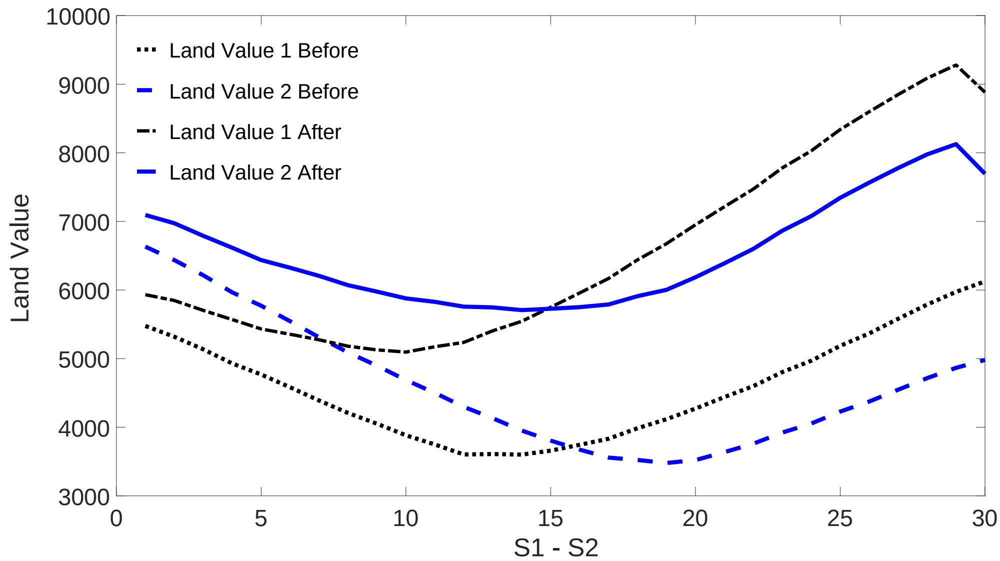

FIGURE 4: Land Values Before and After Change in Volatility (Model 3)

Figure 4 shows the diagonal of land value 1 before change in volatility with dot curve, land value 1 after change in volatility with dash-dot curve, land value 2 before change in volatility with dash curve, and finally land value 2 after change in volatility with solid curve. We plot all the curves in Figure 4 based on Model 3. An increase in volatility shifts the curve to north-west and offers more option value. Higher volatility results in more uncertainty but at the same time adds convexity to the curve that provides valuable option. In Figure 4, we observe that the right leg of curve gains more option value than the left leg when the volatility goes up. Higher option value shifts the optimal exercise frontiers further apart (see Figure 1) and farmers prefers to sit on a more valuable option than exercise it. Overall, an increase in volatility leads to two effects: First, it increases the chance of finding ourself in a situation of optionality to switch and secondly, it moves the exercise frontier further out and lesser the chance of switching.

Scenario 2.B in Table 4 reports the results of a change in drift. When the drift goes up the mean of switch, probability of one or more switch, and probability of two or more switch in Model 1 increase whereas they decrease in Model 2 and Model 3. One explanation is that when drift goes up farmers will gain from the crop returns. Therefore, the growth rate might be enough to cover cash cost in Model 1 and convince farmers to switch but it might not be enough to cover both cash cost and time cost in Model 2 and Model 3. Our results across models display that the mean of switch, probability of one or more switch, and probability of two or more switch between Model 1 and Model 2 as well as between Model 2 and Model 3 is significantly different at the 1% level. However, the difference between Model 1 and Model 2 is larger than the difference between Model 2 and Model 3. The reason refers to have only cash cost in Model 1 but both cash cost and time cost in Model 2 and Model 3.

4.2.3 Discount Rate Sensitivity Analysis

In this section, we examine the impact of an increase in discount rate on the frequency of switching by farmers. Therefore, we increase the discount rate from 5% to 12% and compare the results with our benchmark and across three distinguished models. To incorporate more price risk, we decide to increase the discount rate from 5% to 12%. Hence, the discount rate in our set up should be greater than drift ($r > \mu$) otherwise the denominator of our discount term in an analogous with Gordon model ($PV = c/(r - \mu)$) will be zero or negative. Table 5 presents the results of change in switch frequency by farmers when we increase the discount rate.

Table 5 shows that when discount rate goes up the mean of switch, probability of one or more switch, and probability of two or more switch decrease in all three models. Change in discount rate like volatility has two sides and the outcome depends on how

TABLE 5: *Discount Rate Switching Analysis*

Note 1: *This table presents the mean of switch, the probability of one or more switch, and two or more switch based on 65000 simulation paths over 200 years. The superscripts ***, **, and * indicate significance at the 1%, 5%, and 10% level respectively.*

Variable	**Model 1**	**Model 2**	**Model 3**
Baseline Parameters Analysis			
Mean	1.6049	1.0072***	0.9621***
P ($n \geqslant 1$)	0.8233	0.7551***	0.7425***
P ($n \geqslant 2$)	0.3914	0.2011***	0.1804***
Scenario 3: Change in Discount Rate			
Mean	1.5366	0.9328***	0.6283***
P ($n \geqslant 1$)	0.8124	0.7350***	0.5739***
P ($n \geqslant 2$)	0.3700	0.1661***	0.0510***

large is the change. On the one hand, when discount rate goes up the time value of the option regardless of the option type goes up and farmers prefer to wait and see what will happen. On the other hand, when the discount rate is really high farmers become impatient and prefers to pull the trigger and switch to the profitable crop. In other words, the value of future cash flows may not be worth as much as before and farmers prefer to switch as soon as they get an opportunity that is profitable rather than waiting around and see if they can achieve a higher profit. Our results suggest that an increase in discount rate has a negative impact on switch frequency and farmers prefer to wait more than to switch into alternative crop quickly.

Comparing our results across models show that the mean of switch, probability of one or more switch, and probability of two or more switch decrease from Model 1 to Model 2 and from Model 2 to Model 3. The difference between pair models is significant at the 1% level. We observe that the mean of switch and probability of two or more switch has a larger difference between Model 1 and Model 2 than between Model 2 and Model 3 whereas the probability of one or more switch has a larger difference between Model 2 and Model 3 than between Model 1 and Model 2. In other words, an increase in discount rate causes the farmers to have more multiple switch in Model 2 than Model 1 and they prefer to have no switch or one switch in Model 3 than Model 2. However, the difference between our models almost follow the same pattern we observed in other scenarios.

4.2.4 Correlation Sensitivity Analysis

In this section, we examine the change in correlation between two competing crop prices. We evaluate the switch frequency when correlation between crop prices changes from +0.3 to +0.8 or -0.8. We report our results of change in correlation in Table 6.

TABLE 6: *Correlation Switching Analysis*

***Note 1**: This table presents the mean of switch, the probability of one or more switch, and two or more switch based on 65000 simulation paths over 200 years. The superscripts ***, **, and * indicate significance at the 1%, 5%, and 10% level respectively.*

Variable	**Model 1**	**Model 2**	**Model 3**
Baseline Parameters Analysis			
Mean	1.6049	1.0072***	0.9621***
P ($n \geqslant 1$)	0.8233	0.7551***	0.7425***
P ($n \geqslant 2$)	0.3914	0.2011***	0.1804***
Scenario 4.A: Change in Correlation (Positive)			
Mean	1.0568	0.5076***	0.4378***
P ($n \geqslant 1$)	0.7921	0.4990***	0.4334***
P ($n \geqslant 2$)	0.1863	0.0085***	0.0044***
Scenario 4.B: Change in Correlation (Negative)			
Mean	5.5017	2.6163***	2.5173***
P ($n \geqslant 1$)	0.9468	0.9052***	0.9060
P ($n \geqslant 2$)	0.7962	0.6281***	0.6150***

Table 6 reports the results of our benchmark, Scenario 4.A as a change in correlation in positive direction, and Scenario 4.B as a change in correlation in negative direction. Scenario 4.A shows that all three measures decrease when correlation goes up in positive direction. We postulate that the cash flows from crops move together in the same direction and farmers prefer to stay in the existing state of activity than to switch. Evaluating the results across our models shows that the difference between Model 1 and Model 2 is larger than the difference between Model 2 and Model 3 and all our results are significant at the 1% level.

Scenario 4.B demonstrates that the mean of switch, probability of one or more switch, and probability of two or more switch increase when the correlation goes up in the negative direction. In this case, the cash flows from crops in this scenario move in the opposite direction and when one crop becomes less profitable the competing crop becomes more lucrative. Our results across models demonstrate that the difference between Model 1 and Model 2 is larger than the difference between Model 2 and Model 3 and in all cases decrease. Our results for the difference across pair models are significant at the 1% level except for the probability of one or more switch between Model 2 and Model 3.

5 Conclusion

Our results develop a framework for analysing the optimal land use by farmers. Farmers in our model change their mind of planting one crop as the return of competing crop changes over time. We set up a Black-Scholes partial differential equation model and then solve it with alternative direction implicit approach. Switching crops has the waiting cost and farmers miss their cash flow in this period. Our model provides an optimal solution for this and other possible situations with breaking down the set up into three evolutionary models.

Model 1 is what we see in traditional literature that evaluates a situation where farmers pay a cash cost and switch to a new state of activity. Model 2 is a situation where farmers has to pay the cash cost but they take into account the opportunity cost and missing the cash flow from existing crop. So, farmers approximately estimate the time cost and add it to their cash cost and switch to a new state of activity. Model 3 is the advanced model that is close to what we see in real world. In this model, we examine the time effect very carefully and find the exact time cost. Farmers in this model should wait for a while in "Fallow" state before they become active in a new state of activity. However, farmers have this option to abort the process and go back to the existing crop.

Our study shows that Model 1 that is used through the literature underestimates the switch frequency by farmers. Model 2 presents better results compared with Model 1 but underperform compared with Model 3. The sensitivity results from Model 3 is significantly different from Model 2 at the 1% level in all scenarios except in the case of change in volatility. Our results indicate that an increase in cost, time, volatility, drift, and discount rate has a negative impact on the switch frequency. The change in correlation has a negative impact on switch frequency if it is in the positive direction but a positive impact in the negative direction.

In this study, our model demonstrates the ability to determine which factor has a bigger impact depending on the type of real options. In other words, our model is able to compute the optimal switching according to the type of crop or industry by changing corresponding parameters. Our model shows that how the application of real options will be different in terms of set up. In the practice of real options theory, mostly people are using Model 1 and to lesser extend Model 2 and they might be very surprised to see people do not switch that much. Our set up for Model 3 provides a better explanation for the real world switching problem because it not only considers cash cost and time cost but also it measures the uncertainty about where the crop prices are by the time farmers finish transition. However, it is partly offset by nobility to terminate or abort the switching and go back to what farmers were growing.

The results of this study are important and useful in theory and practice. In theory, we contribute to the literature of valuation especially real option approach by presenting a noble formulation of option to switch. Therefore, researchers scholars, or research institutions and organizations like Ministry for Primary Industry can use the results of this paper for their analysis or develop the idea for further exploration. In practice, our results are helpful for farmers in the first place, as we examined an option situation in agriculture industry. However, practitioners, policy makers, analysts, and advisors in industries with the same nature can utilize our results to optimize their strand of application, develop their policy making process, or achieve effective consultation outcomes.

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